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WALDHAUSEN'S THEORY OF k-FOLD END STRUCTURES: A SURVEY

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Let R^+ be the space of nonnegative real numbers. F. Waldhausen defines a k-fold end structure on a space X as an ordered k-tuple of continuous maps $x_j: X \to R^+$, $1 \leq j \leq k$, yielding a proper map $x: X \to (R^+)^k$. The pairs (X, x) are made into the category E^k of spaces with k-fold end structure. Attachments and expansions in E^k are defined by induction on k, where elementary attachments and expansions in E^0 have their usual meaning. The category E^k/Z consists of objects (X, i) where $i: Z \to X$ is an inclusion in E^k with an attachment of $i(Z)$ to X, and the category $E^k/\!/Z$ consists of pairs (X, i) of E^k/Z that admit retractions $X \to Z$. An infinite complex over Z is a sequence $X = \{X_1 \subset X_2 \subset \cdots \subset X_n \cdots\}$ of inclusions in $E^k/\!/Z$. The abelian group $S_0(Z)$ is then defined as the set of equivalence classes of infinite complexes dominated by finite ones, where the equivalence relation is generated by homotopy equivalence and finite attachment; and the abelian group $S_1(Z)$ is defined as the set of equivalence classes of X, where $X \in E^k/Z$ deformation retracts to Z. The group operations are gluing over Z. This paper presents the Waldhausen theory with some additions and in particular the proof of Waldhausen's proposition that there exists a natural exact sequence $0 \to S_1(Z \times R) \xrightarrow{\partial} S_0(Z)$ by utilizing methods of L.C. Siebenmann. Waldhausen developed this theory while seeking to prove the topological invariance of Whitehead torsion; however, the end structures also have application in studying the splitting of a noncompact manifold as a product with R [1].

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k-fold end structures	generalizations of attachments and expansions
k-tuples of continuous	small cell property
non-compact manifolds	obstruction to splitting

1. Introduction

Let R^+ be the euclidean half line. F. Waldhausen defines a k-fold end structure on a space X as an ordered k-tuple of continuous maps $x_j: X \to R^+$, $1 \leq j \leq k$, yielding a proper map $x: X \to (R^+)^k$. The pairs (X, x) are made into the category E^k of spaces with k-fold end structure. Attachments and expansions in E^k are defined by induction on k, where elementary attachments and expansions in E^0 have their usual meaning. The category E^k/Z consists of objects (X, i) where $i: Z \to X$ is an inclusion in E^k with an attachment of $i(Z)$ to X, and the category

$E^k/\!/Z$ consists of pairs (X, i) of E^k/Z that admit retractions $X \to Z$. An infinite complex over Z is a sequence $X = \{X_1 \subset X_2 \subset \cdots \subset X_n \cdots\}$ of inclusions in $E^k/\!/Z$. The abelian group $S_0(Z)$ is then defined as the set of equivalence classes of infinite complexes dominated by finite ones, where the equivalence relation is generated by homotopy equivalence and finite attachment; and the abelian group $S_1(Z)$ is defined as the set of equivalence classes of X, where $X \in E^k/Z$ deformation retracts to Z. The group operations are gluing over Z.

The present work proves the proposition of Waldhausen that for $Z \in E^{k-1}$ there exists a natural exact sequence $0 \to S_1(Z \times R) \xrightarrow{\sigma} S_0(Z)$ by utilizing methods of Siebenmann. It is shown that, given a proper homotopy equivalence $X \to Z \times R$, an element of $S_1(Z \times R)$ is generated, the image of which under σ is the obstruction to the splitting of X as a CW-complex. Sum theorems for the S_0 and S_1 groups are also proven, and the concept of connectedness in E^k is added to the concepts of Waldhausen

2. Spaces with multiple end structures

Let R^+ be the space of nonnegative real numbers. F. Waldhausen [5] defines a *k-fold end structure* on a space X as an ordered k-tuple of continuous maps $x_j : X \to R^+$, $1 \leq j \leq k$, yielding a map $x : X \to (R^+)^k$. Such an end structure is said to be *locally compact* if X is locally compact and x is proper (i.e., inverse images of compact sets are compact). Note that a locally compact space may always be given a 1-fold end structure. The following definitions concerning k-fold end structures are also due to Waldhausen [5].

An equivalence relation on k-fold end structures on X is defined by majorization. For two k-fold end structures on X, x and x', x is *majorized* by x' when for any j, $1 \leq j \leq k$, and any tuple $n_j, n_{j+1}, \ldots, n_k$, there exists n'_j such that for arbitrary $a \in X$ if $x_i(a) \leq n_i$, $j \leq i \leq k$, then $x'_j(a) \leq n'_j$. Two k-fold end structures on X are equivalent if either is majorized by the other. For example, any two locally compact 1-fold end structures on X are equivalent.

A pair (X, x) is defined to be a *space with k-fold end structure*, and these spaces are made into a category E^k by defining a map $f : (X, x) \to (Y, y)$ to be a continuous map $f : X \to Y$ with the property that the end structures $y \circ f$ and x on X are equivalent. If f is also a homeomorphism then $f : (X, x) \to (Y, y)$ is called an *isomorphism* in E^k. Note that if $f : (X, x) \to (Y, y)$ is a map in E^k where (X, x) and (Y, y) have locally compact end structures then f is proper; however, the converse need not be true.

For N a countable, discrete set, there exists the *infinite disjoint union functor* $\perp : (E^{k-1})^N \to E^k$, which is well defined up to isomorphism. If a space in E^k is isomorphic to a space in the image of $\perp$, it is said to be a *disjoint union* of spaces in E^{k-1} (compare the geometric diagram below corresponding to a 'bumpy pushout diagram'). A map in E^k being a disjoint union of maps in E^{k-1} is similarly defined.

Homotopies in E^k are defined by a straightforward extension. A map $F:(X,x)\times[0,1]\to(Y,y)$ in E^k (where $(X,x)\times[0,1]$ is given the product k-fold end structure induced from the projection $p_1:X\times[0,1]\to X$) is defined to be a homotopy in E^k from $f_0:(X,x)\to(Y,y)$ to $f_1:(X,x)\to(Y,y)$ whenever the f_i are given by the restrictions $F|_{(X,x)\times i}$, $i=0,1$.

A map $f:(X,x)\to(Y,y)$ in E^k is called an inclusion if $f:X\to Y$ is itself an inclusion. It is assumed that the homotopy extension property holds in E^k, that 'pushouts' can be formed, and consequently that inclusions are in the category of underlying spaces of E^k. An inclusion $f:(X,x)\to(Y,y)$ in E^k is said to be *bumpy* if the following pushout diagram of inclusions in E^k exists such that f' is a disjoint union of inclusions in E^{k-1}.

$$\begin{tikzcd}
(X,x) \arrow[r, "f"] & (Y,y) \\
(X',x') \arrow[u] \arrow[r, "f'"] & (Y',y') \arrow[u]
\end{tikzcd}$$

To illustrate these concepts by a simple geometrical example, consider the case when X is the real line, Y is X with unit intervals attached by one end at each integer in X, X' is the set of points in X corresponding to the integers, and Y' is the countable collection of unit intervals. This then yields the following geometrical diagram (Fig. 1) that corresponds to the pushout diagram of above.

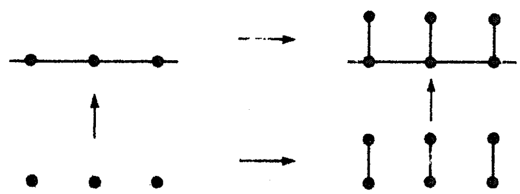

Fig. 1.

An attachment in E^k is defined by induction on k:

(i) an elementary attachment in E^0 has its usual meaning (attaching a cell by a cellular attaching map defined on its boundary);

(ii) an attachment in E^k is a finite composition of elementary attachments; and

(iii) an elementary attachment in E^k is a bumpy inclusion obtained from a pushout diagram as above with f' as a disjoint union of attachments in E^{k-1}.

By a similar induction on k, an expansion in E^k is defined:

(i) an elementary expansion in E^0 has its usual definition (for an n-ball B^n and K in E^0, consider a cellular map $B^n\to K$ and take its mapping cylinder relative to the boundary of the ball – see [2] for an equivalent definition and analogs in E^0);

(ii) an expansion in E^k is a finite composition of elementary expansions in E^k; and

(iii) an elementary expansion in E^k is a bumpy inclusion obtained from a pushout diagram as above with f' as a disjoint union of expansions in E^{k-1}.

Note as usual that an expansion is an attachment, but that the converse need not hold.

Suppose for $(X, x) \in E^k$ that the space X has a locally finite cellular structure. Then (X, x) is said to have the *small cell property* if given j, $1 \leq j \leq k$, and given positive numbers $n_j, n_{j+1}, \ldots, n_k$, there exists a positive number n_j' such that for an arbitrary cell C in X with $x_i(C) \subset [0, n_i]$, $j+1 \leq i \leq k$, and with $x_j(C) \cap [0, n_j] \neq \emptyset$, it follows that $x_j(C) \subset [0, n_j']$. By way of application, consider a finite dimensional, locally finite CW-complex X with $(X, x) \in E^k$. It follows that there is an attachment from $\emptyset$ to X if and only if (X, x) has the small cell property. The following example, suggested by Waldhausen indicates the type of problem that may arise.

Example 2.1. Let $Y = R^2$ and obtain X from Y by attaching unit intervals I_n at the points $(n, 0)$ where n is a positive integer (see Fig. 2). Define the map $f: X \to Y$ by mapping I_n around the circle $C(0, n)$, centered at the origin with radius of n.

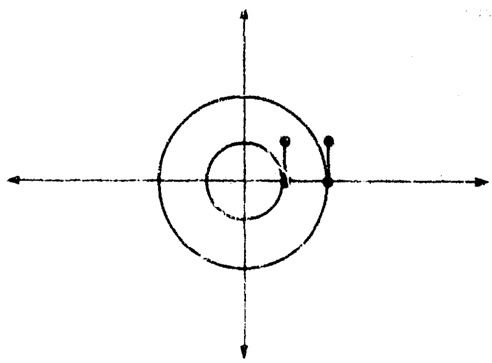

Fig. 2.

Let $y: Y \to R^+$ be a 1-fold end structure on Y generated by $(a, b) \mapsto (a^2 + b^2)^{1/2}$, and set $x = y \circ f: X \to R^+$ as a 1-fold end structure on X. Then $(X, x) \in E^1$, and the small cell property is satisfied. On the other hand, consider the case when $y: Y \to (R^+)^2$ is the 2-fold end structure on Y defined by $(a, b) \mapsto (|a|, |b|)$, again letting $x = y \circ f: X \to (R^+)^2$. Then $(X, x) \in E^2$, but the definition of the small cell property is not satisfied for $j = 2$.

For $Z \in E^k$, let E^k/Z denote the category of pairs (X, i) where $i: Z \to X$ is an inclusion in E^k such that there exists an attachment from $i(Z)$ to X. A map $f: (X, i) \to (X', i')$ in the category E^k/Z is a map $f: X \to X'$ satisfying $f \circ i = i'$. Let $E^k /\!/ Z$ denote the category of triples (X, i, r) where (X, i) is an object of E^k/Z and $r: X \to Z$ is a retraction ($r \circ i = \mathrm{id}|_Z$). A map $f: (X, i, r) \to (X', i', r')$ in the category

$E^k/\!/Z$ is a map f in E^k/Z with the property that $f \circ r \simeq r' \circ f$ where the homotopy is in $i(Z)$. Such an f is an inclusion in $E^k/\!/Z$ if $f: X \to X'$ is an inclusion and if $f \circ r = r' \circ f$.

Consider now a map $g: Z \to Z'$ where Z and Z' are in E^k. The map g induces the base change functor $g_*: E^k/Z \to E^k/Z'$, where g_* sends (X, i) to (X', i') with X' being the quotient space $(X \cup Z')/\{a = g(a) \mid a \in Z\}$ and i' being induced from $Z' \to X \cup Z'$. Given a map $f: (X, i) \to (Y, j)$ in E^k/Z, then $g_*(f)$ is the map generated from $f \cup \mathrm{id}: X \cup Z' \to Y \cup Z'$. Similarly, g induces the base change functor $g_*: E^k/\!/Z \to E^k/\!/Z'$.

The notion of simple homotopy types is now defined. For (X, i) and (Y, j) in E^k/Z, the two spaces X and Y are said to be *simply equivalent* if there exist expansions from X to X' and from Y to Y' and an isomorphism in E^k/Z from X' to Y'. Since expansions and isomorphisms admit pushout diagrams, this yields an equivalence relation, and $[X]$ will denote the equivalence class of X. Let $S_1(Z)$ denote the set of simple homotopy types on Z, i.e., the set of equivalence classes $[X]$ represented by the $X \in E^k/Z$ that deformation retract to Z. By defining on $S_1(Z)$ the operation of gluing over the base Z, Waldhausen demonstrates that $S_1(Z)$ becomes an abelian group [5]. An inverse for a given $[X]$ is produced by considering a homotopy $F: X \times I \to X$ with $F_0 = \mathrm{id}_X$ and $F_1(X) \subset Z$. Let M_1 be the mapping cylinder of F_1. Note that M_1 represents 0 since there is an expansion from Z to M_1. Letting $\perp$ denote the operation of gluing and considering $F_{1*}(M_1)$, it follows that $[X \perp F_{1*}(M_1)] = [M_1] = 0$. Hence $[F_{1*}(M_1)]$ is the desired inverse of $[X]$ (see [5] and compare Theorem 6.1 of [2]).

Now define an *infinite complex* over $Z \in E_k$ as a possibly infinite sequence of inclusions in $E^k/\!/Z$ of the form $X = \{X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots\}$. If X and Y are two such infinite complexes over Z, then a map from X to Y is defined as a compatible system of maps

$$f = \{f_1, f_2, \ldots\}, \qquad f_j \in \varinjlim_i \mathrm{Hom}(X_j, Y_i).$$

This signifies that of the following two diagrams, the first commutes up to homotopy if one goes 'far enough out' in the diagram, and the second commutes up to homotopy.

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & X_3 \arrow[r] \arrow[d, "f_3"'] & \cdots \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & \cdots \end{tikzcd} \qquad \begin{tikzcd} X_j \arrow[rr] \arrow[dr, "f_j"'] & & X_{j+1} \arrow[dl, "f_{j+1}"] \\ & \varinjlim_i Y_i & \end{tikzcd}$$

sequences) generate a subcategory equivalent to $E^k /\!/ Z$. Consider the following diagram where the pair (K, L) is equivalent to a pair of $E^k /\!/ Z$, X and Y are from $\mathrm{Inf}(E^k /\!/ Z)$, and the diagram is a pushout in $\mathrm{Inf}(E^k /\!/ Z)$.

$$\begin{array}{ccc} K & \dashrightarrow & Y \\ \cup & & \cup \\ L & \longrightarrow & X \end{array}$$

If after appropriate deformations of the structural retractions such a diagram exists such that all the maps commute with the structural retractions, then there is a *finite attachment* from X to Y. The following geometric diagram (Fig. 3) illustrates these concepts and corresponds to the pushout diagram of above.

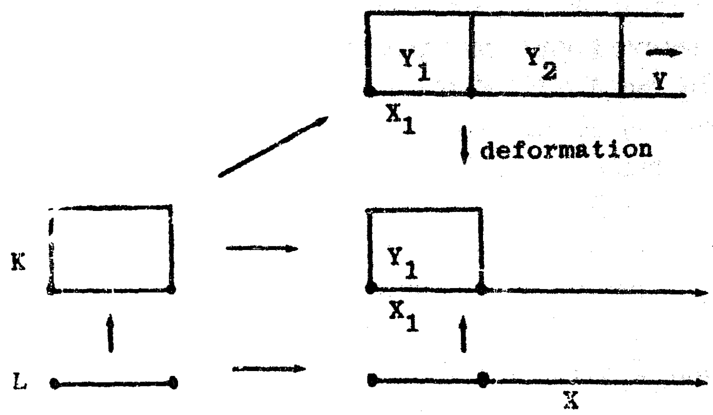

Fig. 3.

Now $X \in \mathrm{Inf}(E^k /\!/ Z)$ is *dominated* by $X' \in \mathrm{Inf}(E^k /\!/ Z)$ if there exist maps $f: X \to X'$ and $g: X' \to X$ such that $g \circ f \simeq \mathrm{id}\, X$. If there exists $X' \in E^k /\!/ Z$ which dominates X, then X is said to be dominated by a finite complex. Let $S_0(Z)$ denote the set of equivalence classes of infinite complexes over Z that are dominated by finite ones, where the equivalence relation is generated by homotopy equivalence and finite attachment. Again defining the operation on $S_0(Z)$ as gluing over the base Z induces the algebraic structure of an abelian group. The inverse of a given $[X] \in S_0(Z)$ is $[\Sigma(X)] \in S_0(Z)$ where $\Sigma(X)$ is the suspension of X defined by $\Sigma(X) = (X \times_Z I)/(X \times_Z \partial I)$ (see [5]).

3. Sum theorems

Sum theorems for the groups $S_0(Z)$ and $S_1(Z)$ will now be proven. First consider $A, B, C \in E^k$ and inclusions $A \xrightarrow{f} B \to C$, where $(C, A) \in S_1(A)$, $(B, A) \in S_1(A)$, and

$(C, B) \in S_1(B)$. Let $[C, A]$ denote $f_*(C, A) = (C \cup_A B, B)$, where $f_*: E^k/A \to E^k/B$ is the induced 'change of base' functor. Note that $[B, A] = f_*(B, A) = (B \cup_A B, B)$ and $[C, B] = \mathrm{id}_*(C, B) = (C \cup_B B, B) = (C, B)$. It now follows from $(C \cup_A B, B) = (B \cup_A B, B) \cup (C, B)$ that $[C, A] = [B, A] + [C, B]$. This establishes the following lemma (compare formula (1.1) of [4]).

Lemma 3.1. *If $A \xrightarrow{f} B \to C$ are inclusions in E^k with (C, A), $(B, A) \in E^k/A$ and $(C, B) \in E^k/B$, then $[C, A] = [B, A] + [C, B]$.*

In the above setting, let C be an infinite sequence over A and over B and B an infinite sequence over A, and let $[C, A]$ denote the equivalence class in $S_0(B)$ of $f_*(C, A)$, $f_*: \mathrm{Inf}(E^k /\!/ A) \to \mathrm{Inf}(E^k /\!/ B)$. Analogously to the above argument, Lemma 3.2 follows.

Lemma 3.2. *If $A \xrightarrow{f} B \to C$ are inclusions with $C \in \mathrm{Inf}(E^k /\!/ A)$, $C \in \mathrm{Inf}(E^k /\!/ B)$, and $B \in \mathrm{Inf}(E^k /\!/ A)$, then $[C, A] = [B, A] + [C, B]$.*

A sum theorem for $S_1(Z)$ is now established.

Theorem 3.3. *Let $[X] \in S_1(Z)$ be represented by (X, Z). Suppose $(X, Z) = (X_1, Z_1) \cup (X_2, Z_2)$ and $(X_1, Z_1) \cap (X_2, Z_2) = (X_0, Z_0)$ with $[X_0] \in S_1(Z_0)$, $[X_1] \in S_1(Z_1)$, $[X_2] \in S_1(Z_2)$, $f_i: Z_i \to X_i$, and $g_i: Z_i \to Z$, $i = 0, 1, 2$. Then*

$$(X, Z) = g_{1*}(X_1, Z_1) + g_{2*}(X_2, Z_2) - g_{0*}(X_0, Z_0).$$

Proof. Consider the following diagram of inclusions

$$\begin{tikzcd} A \arrow[r] & C = A \cup B \\ A \cap B \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

Note that pushouts are in the equivalent relation of $S_1(B)$, i.e., for $(C, B) \in S_1(B)$,

$$[C, B] = (C, B) = (A \underset{A \cap B}{\cup} B, B) = [A, A \cap B].$$

This establishes a type of 'excision' theorem for S_1 groups (compare the 'pushouts' on p. 481 of [4]).

Applications of Lemma 3.1 and the above excision process yield the subsequent chain of equalities (compare the derivation of sum formulas in [4], [3], and [2]).

$$
\begin{aligned}
(X, Z) &= [X, Z] \\
&= [X_1 Z \cup X_1] + [Z \cup X_1, Z] \\
&= [X_2, Z_2 \cup X_0] + [X_1, Z_1] \\
&= [X_2, Z_2] - [Z_2 \cup X_0, Z_2] + [X_1, Z_1] \\
&= [X_1, Z_1] + [X_2, Z_2] - [X_0, Z_0] \\
&= g_{1*}(X_1, Z_1) + g_{2*}(X_2, Z_2) - g_{0*}(X_0, Z_0).
\end{aligned}
$$

Theorem 3.4 (Sum theorem, $S_0(Z)$). *Let $[X] \in S_0(Z)$ be represented by (X, Z), where $X = \{X_1 \subset X_2 \subset X_3 \subset \cdots\}$, X dominated by some finite $K = \{K_1 \subset K_2 \subset \cdots\}$, $\alpha : X \to K, \beta : K \to X, \beta \circ \alpha = \mathrm{id}\, X$. Suppose $(X, Z) = (X^1, Z^1) \cup (X^2, Z^2), (X^1, Z^1) \cap (X^2, Z^2) = (X^0, Z^0)$, and $g_i : Z^i \to Z$. Show $(X, Z) = g_{1*}(X^1, Z^1) + g_{2*}(X^2, Z^2) - g_{0*}(X^0, Z^0)$, when $[X^1] \in S_0(Z^1)$, $[X^2] \in S_0(Z^2)$, and $[X^0] \in S_0(Z^0)$.*

Proof. Note that the equivalence relation of S_0 groups (generated by homotopy equivalence and finite attachment) admits pushout diagrams. To see this, consider A, B, $A \cup B \in E^k$, $[U] \in S_0(A)$, $[U \cup V] \in S_0(A \cup B)$ and the following diagram.

$$
\begin{tikzcd}
A \cup B \arrow[r] & U \cup V \\
A \arrow[u] \arrow[r] & U \arrow[u]
\end{tikzcd}
$$

This diagram yields a finite attachment from U to $U \cup V$, and consequently $[U \cup V, A \cup B] = [U, A]$ in $S_0(A \cup B)$. This establishes a type of 'excision' theorem for S_0 groups. The desired sum theorem now follows as in Theorem 3.3 by application of Lemma 3.2 and the excision process.

4. Some exact sequences

Consider now the inclusion $(R^+)^{k-1} \to (R^+)^{k-1} \times 0 \subset (R^+)^k$ that produces a functor and embedding $e_k : E^{k-1} \to E^k$. Utilizing e_k and considering Z an element of E^k, let $(E^{k-1} \to Z)$ denote the category of maps $Z' \to Z$ with $Z' \in E^{k-1}$. Now let ${}^{b}E^k/Z$ denote the subcategory of those objects in E^k/Z that can be generated from E^{k-1}/Z' by a base change $Z' \to Z$, $Z' \in E^{k-1}$. The concepts and theories of simple homotopy types and domination may be constructed for ${}^{b}E^k/Z$ just as for E^k/Z, yielding the corresponding abelian groups $S_1^{b}(Z)$ and $S_0^{b}(Z)$. For example, if $Z' \xrightarrow{f} Z$ with $[X'] \in S_1(Z')$ then $[X' \cup_f Z]$ is the corresponding $[X] \in S_1(Z)$.

For an inverse system $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \cdots$, the maps may be viewed via their product

$$\prod_{j\geq 1} G_j \xrightarrow{d} G_0 \times \prod_{j\geq 1} G_j = \prod_{j\geq 0} G_j.$$

Then letting $i: \prod_{j\geq 1} G_j \to \prod_{j\geq 0} G_j$, denote the natural injection, the map $i-d$: $\prod_{j\geq 1} G_j \to \prod_{j\geq 0} G_j$ follows. Using this map $i-d$, define the *inverse limit* of the system as $\lim G_j = \ker(i-d)$ and the *derived* of the inverse limit as $\lim' G_j = \operatorname{coker}(i-d)$. For the base space $Z \in E^k$, let Z_i be a system of *neighborhoods of infinity in direction k* where such a system is defined by a nondecreasing sequence of $r_i \in R^+$ such that $z_k(Z_i) \subset [r_i, \infty)$ and $z_k^{-1}([r_i, \infty)) \subset Z_{i-1}$. Such a system of neighborhoods produces the inverse systems $S_1^b(Z_0) \leftarrow S_1^b(Z_1) \leftarrow S_1^b(Z_1) \leftarrow \cdots$ and $S_0^b(Z_0) \leftarrow S_0^b(Z_1) \leftarrow S_0^b(Z_2) \leftarrow \cdots$ and sets the stage for the following theorem.

Theorem 4.1 (Waldhausen). *For $Z \in E^k$ and $\{Z_i\}$ a system of neighborhoods of infinity in direction k, the following sequence is exact.*

$$0 \to \lim' S_1^b(Z_i) \xrightarrow{\sigma_1} S_1(Z) \xrightarrow{\sigma_0} \lim S_0^b(Z_i) \to 0.$$

Proof. The proof follows essentially the arguments of Siebenmann in [4] and utilizes strongly the infinite disjoint union functor that allows one to view 'finite' objects along the k-axis in E^k (i.e., objects A that are finite in direction k: there exists $r'_k \in R^+$ such that $z_k(A) \leq r'_k$).

Let $[X] \in S_1(Z)$. Then X is a CW-complex over Z that deformation retracts to Z, and $(X, x) \in E^k$ where $x_i = z_i \circ r: X \to Z \to R^+$ with $r: X \to Z$ the guaranteed retraction. For a given Z_i, recall that $z_k^{-1}([r_{i+1}, \infty)) \subset Z_i$, where the r_i are from the definition of a system of neighborhoods of ∞ in direction k. Let $X_i = x_k^{-1}([r_{i+1}, \infty))$. Then

$$r(X_i) = r(x_k^{-1}([r_{i+1}, \infty)) = r \circ r^{-1} \circ z_k^{-1}([r_{i+1}, \infty)) \subset Z_i.$$

Assuming X_i is a subcomplex of X with this property, it follows that for a given Z_i there exists a subcomplex X_i of X such that X_i deformation retracts inside X into Z_i and such that $x_k^{-1}([r'_i, \infty)) \subset X_i$ for some $r'_i \subset R^+$, $r'_i \geq r_i$. (See Fig. 4.) Hence

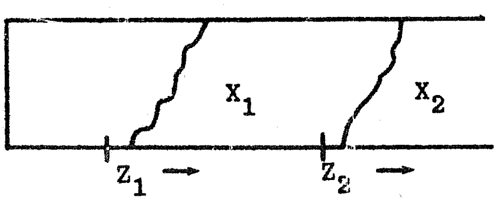

Fig. 4.

for some $j > i$, it may be assumed that Z_j is chosen such that the corresponding X_j deformation retracts inside X_i onto Z_j. Let

$$W_1 = x_k^{-1}([r_{i+1}, r']), \quad r_{i+1} < r' < r_j,$$

and let W_2 equal the closure of $X_i-(W_1\cup X_j)$. Consider id_{W_1} and the deformation of X restricted to X_j. Applying the homotopy extension property to a combination of these two maps over W_2 yields an X_i' finite in direction k (i.e., $x_k(X_i')\subset[0,r_i'']$ for some $r_i''\in R^+$) such that X_i deformation retracts inside itself into $Z_i\cup X_i'$. (See Figs. 5 and 6.)

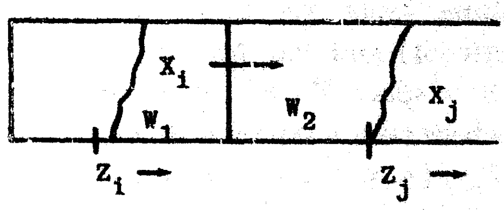

Fig. 5.

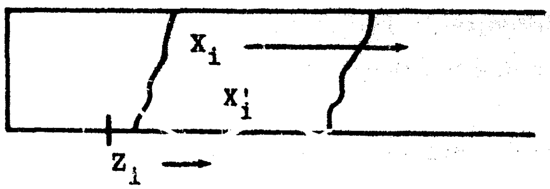

Fig. 6.

Assume now that each Z_i has a corresponding X_i and X_i' such that X_i deformation retracts inside itself into $Z_i\cup X_i'$ (this assumption is permissible since one may always pass to a subsequence of the Z_i with the desired property). This then yields for each Z_i a corresponding element in $S_0^b(Z_i)$ and constructs a map $\sigma_0: S_1(Z)\to \lim S_0^b(Z_i)$. Note that σ_0 is well defined since two representatives of an element of $S_1(Z)$ expand and collapse to another representative that may be used in the construction and since one may pass to a refining subsequence of two given subsequences of X_i' in the construction above.

Suppose that $\sigma_0([X])=0$. Then for each i, X_i represents 0 as a domination problem over Z_i, and after suitable expansions on X, X_i is homotopy equivalent as an infinite complex over Z to the finite complex generated from $Z\cup X_i'$. In particular, $Z_i\cup X_i'\to X_i$ is a homotopy equivalence. Comparable to Siebenmann's terminology in [4], call such a subcomplex as X_i' a *side h-section* of (X_i, Z_i). The object now is to apply Siebenmann's methods in [4] that generate a *flat h-section* from side h-sections, i.e., the goal is to perform an elementary expansion on X to make X_i contractible inside of itself to Z_i. Note that a disjoint union along the k-axis of constructions that are finite in direction k make sense in E^k. Hence the above process may be applied to all X_i.

Since X_i' is finite in direction k, there exists $j>i$ such that $X_j\subset X_i-X_i'$. Set $W=(X_i-X_j)\cup X_i'\cup X_j'$ and $V=W\cap X$ (see Fig. 7). There exists a cellular deformation retraction $q: W\to X_i'\cup Z_i$ such that $q(X_j')\subset Z_i$.

Let $\rho: X_j'\cup Z_j\to Z_i$ be $q|_{X_j'}$ on X_j' and the identity on Z_j. Use ρ to glue X_j to Z_i in forming the space $Z_i\cup_\rho X_j$. Let Y_i be the image of X in the adjunction space,

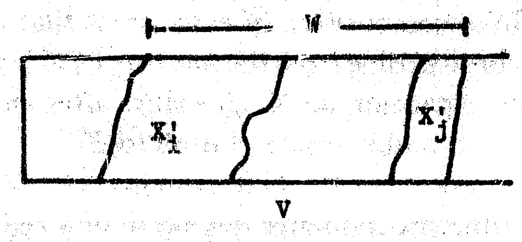

Fig. 7.

thus yielding $Z_i \cup_p X_j = Z_i \cup Y_j$ (see Fig. 8). Note that $i_1 : X_i' \cup Z_i \to X_i' \cup Z_i \cup Y_j$ is a homotopy equivalence, as is i_2: $W \cup Z_i \to X_i$. Consider the retraction $q' : W \cup Z_i \to X_i' \cup Z_i$, extending $q : W \to X_i' \cup V$ and i_2.

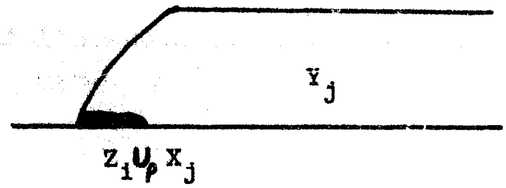

Fig. 8.

Let $[i_1]$ and $[i_2]$ denote $(X_i' \cup Z_i \cup Y_j, X_i' \cup Z_i) \in S_1(X_i' \cup Z_i)$ and $(X_i \cup X_i', W \cup Z_i) \in S_1(W \cup Z_i)$, respectively. Note that

$$q'_*([i_2]) = (X_i \underset{W \cup Z_i}{\cup} M'_{q'}, X_i' \cup Z_i),$$

where $M_{q'}$ is the mapping cylinder of q', which equals $(X_i' \cup Z_i \cup Y_j, X_i' \cup Z_i) = [i_1]$ by collapsing along $M_{q'}$. Hence $q'_* [i_2] = [i_1]$. Write $i_3 : X_i' \cup Z_i \to W \cup Z_i$, and note that $i_{3*} [i_2 \circ i_3] = i_{3*}[i_3] + [i_2]$ by an application of Lemma 3.1. Applying q'_* yields

$$[i_2 \circ i_3] = [i_3] + q'_*[i_2] = [i_3] + [i_1],$$

where $i_2 \circ i_3$: $X_i' \cup Z_i \to X_i \cup X_i'$. Now $[i_3] + [i_1]$ is represented by the sum of $W \cup Z_i$ and $X_i' \cup Z_i \cup Y_j$ with the two copies of $X_i' \cup Z_i$ glued together, which will be denoted by $X_i' \cup Z_i \cup U$. Then (U, Z_i) has the property of possessing the subcomplex $Z_i \cup Y_j$, cofinite in direction k, such that $Z_i \to Z_i \cup Y_j$ is a homotopy equivalence. Again following Siebenmann's terminology in [4], call such a subcomplex a *flat h-section* for (U, Z_i).

Now $[i \circ i_3] = [i_3] + [i_1]$ yields the following commutative diagram as in [7] with s and t as expansions in E^k.

$$\begin{tikzcd}
U \arrow[r, "s"] & S \\
X_i \cup Z_i \arrow[u] \arrow[r] & X_i' \arrow[u, "t"']
\end{tikzcd}$$

Each expansion is a finite composition of expansions that are bumpy. Thus a flat h-section for (S, Z_i) is constructible from the one for (U, Z_i), and then a flat h-section for (X_i, Z_i) after a finite expansion on X. Therefore after an expansion on X, it can be assumed that X_i is contractible inside of itself to Z_i.

A boundary for X_i is any finite X_i'' that contains $X_i \cap \mathrm{cl}(X - X_i)$. The existence of boundaries follows from the inductive definition of a cell complex in E^k/Z. Let B_i be a boundary for X_i. The object now is to make B_i trivial in the sense that it is contractible to $B_i \cap Z_i$ inside of itself by the same deformation that contracts X_i to Z_i. First note the following lemma, which is a slight modification in both statement and proof (the modification being a generalization to finite in direction k) of the lemma on p. 488 of [4].

Lemma 4.2. *Let $M \to N$ be a homotopy equivalence with $N - M$ finite in direction k. Then there exists a subcomplex K of N finite in direction k such that $N - M \subset K$ and $K \cap M \to K$ is a homotopy equivalence.*

Consider the side h-section $X_i' \cup Z_i \to X_i$ and the flat h-section for (X_i, Z_i) as constructed above. Assuming this result for all X_i, there exists $j > i$ such that $X_j \cap X_i' = \emptyset$. Consider $X_i' \cup Z_i \to X_i' \cup Z_i \cup X_j \to X_i$. The first inclusion and the composition are homotopy equivalences. Thus the second inclusion is also a homotopy equivalence, and there exists a homotopy h_t of $\mathrm{id}|_{X_i}$, fixing $X_i' \cup Z_i \cup X_j$, to a retraction onto $X_i' \cup Z_i \cup X_j$. Since $X_i - (X_i' \cup Z_i \cup X_j)$ is finite in direction k, there exists a subcomplex W_1, finite in direction k, of X_i containing $X_i - X_j$ such that, for $V_1 = W_1 \cap Z_i$ and $Y_j = W_1 \cap X_j$, h_t restricts to a deformation of id_{W_1} to a retraction onto $X_i' \cup V_1 \cup Y_j$.

Choose now $m > j$ and its corresponding side h-section $X_m' \cup Z_m \to X_m$ such that $X_m \subset X_j$ and $X_m \cap W_1 = \emptyset$. Set

$$W_2 = (X_j - X_m) \cup X_m', \ V_2 = W_2 \cap Z_i$$

(see Fig. 9). Now $X_j \cap Z_i \to W_2 \cup Z_i$ is a homotopy equivalence since both $W_2 \cup Z_i \to X_j$ and $X_j \cap Z_i \to X_j$ are homotopy equivalences. Lemma 4.2 implies $V_2 \to W_2$ becomes a homotopy equivalence by adding to X_m' (and thus to W_2 and V_2) a finite piece of $X_m \cap Z_i$. Therefore assume that $V_2 \to W_2$ is a homotopy equivalence and that $B_i = W_2$ by translation via the subscripts. It now follows that $(X_i, B_i) \rightsquigarrow$

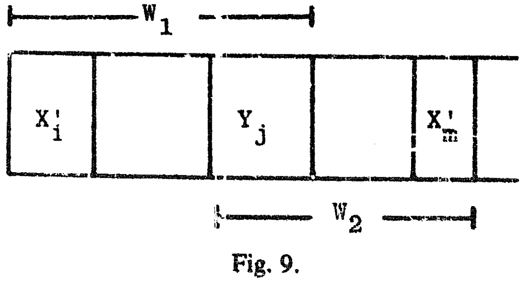

Fig. 9.

$(Z_i, B_i \cap Z_i)$, and B_i is trivial as desired after another elementary expansion. This may be done for any i, and it may be assumed that the entire sequence of Z_i has the property that $B_i \cap B_{i+1} = \emptyset$ (by passing to a subsequence with this desired property).

It now follows that $[X] \in S_1(Z)$ is in the image of $\sigma_1 : \lim' S_1^b(Z_i) \to S_1(Z)$. Hence $\ker \sigma_0 \subset \operatorname{im} \sigma_1$. Note that σ_1 is defined by gluing the components of an element of $\lim' S_1^b(Z_i)$ along the base Z and that the equivalence relation in the definition of $\lim'$ assures that σ_1 is well defined. Now an element of $\operatorname{im} \sigma_1$ being generated by gluing along Z objects that are finite in direction k and that deformation retract to Z guarantees that each element in $S_0^b(Z_i)$ generated by the definition of σ_0 is trivial as a domination problem over Z_i. Hence $\operatorname{im} \sigma_1 \subseteq \ker \sigma_0$ as desired.

To complete the proof of Theorem 4.1, it suffices to show that σ_0 is onto and that σ_1 is 1–1. Let $(A_1, A_2, \ldots) \in \lim S_0^b(Z_i)$, then $A_i \in S_0^b(Z_i)$ where $A_i = \{B_1 \subset B_2 \subset \cdots\}$ with the $B_j \in {}^bE^k/Z_i$. Also, A_i is dominated by $Z_i \cup B$ for some $B \subset \bigcup B_j$ that is finite in direction k, and thus $\bigcup B_j$ deformation retracts into $Z_i \cup B$. In order to be an element in $S_1(Z)$, however, $Z \cup (\bigcup B_j)$ must deformation retract onto Z. Recall that there exists a retraction $r : X_i \to Z_i$, and consider the mapping cylinder $M_r|_B$ (relative Z_i). Attaching $M_r|_B$ to $Z \cup (\bigcup B_j)$ (a finite attachment in E^k) generates a homotopy equivalence $Z_i \to Z_i \cup (\bigcup B_j)$ and yields the desired deformation retraction. To make sure the homotopy equivalence is in the category, the "side h-section, flat h-section, side h-section" argument used above in the application of Lemma 4.2 must be applied. Thus σ_0 is onto.

Consider $(D_1, D_2, \ldots) \in \prod_i S_1^b(Z_i)$ that is in $\ker \sigma_1$. Then in general, $Z \cup (\bigcup D_i) \nearrow G \searrow Z$ in E^k for some G; and to see that $(D_1, D_2, \ldots) = 0$ in $\lim' S_1^b(Z_i)$, it suffices to construct inductively G_i such that $(G_1, G_2, \ldots) \in \prod_i S_1^b(Z_i)$ and $D_i + G_{i+1}$ expands and collapses to G_i. Write the collapse $G \searrow Z$ as a sequence of elementary (bumpy) collapses

$$G = G^0 \searrow G^1 \searrow \cdots \searrow G^m = Z,$$

where $G^i - G^{i+1} = F_1^i \cup F_2^i \cup \cdots$ with the F_j^i as disjoint collapses in E^{k-1}. Note that finitely many complexes of F_1^0, $F_2^0, \ldots$ meet D_j. Partially collapse $G^0 \searrow G_j^1$ (bumpy) where $D_j \subset G_j^1$ and G_j^1 is the smallest subcomplex of G^0 containing $D_j \cup G^1$. Then the bumpy collapse $G_j^1 \searrow G^1$ is finite in direction k. Next obtain a partial collapse from the collapse (bumpy) $G_j^1 \searrow G^1 \searrow G^2$, collapsing across all possible F_t^1 in the second bumpy collapse that leave D_j, thus obtaining G_j^2 where $G^0 \searrow G_j^1 \searrow G_j^2 \searrow G^2$ ($G_j^2 \searrow G^2$ need not be bumpy). Since only finitely many of the F_t^2 can meet either D_j or the F_t^0 still in G_j^1, the collapse $G_j^2 \searrow G^2$ is finite in direction k. Inductively obtain $G_j^3, \ldots, G_j^m$ where $G^0 \searrow G_j^1 \searrow G_j^2 \cdots \searrow G_j^m \searrow Z$. All of these collapses are bumpy except the last, which is obtained by returning to the original sequences of bumpy collapses and doing the collapses that remain. Note the G_j^m is finite in direction k and contains D_j. Set $G_j = G_j^m$. Then $D_j + G_{j+1} \nearrow\searrow G_j$ by first expanding to include G_j and then collapsing, using parts of the bumpy collapses (expansions) in $G \searrow G^1 \searrow \cdots \searrow G^m = Z$.

Note that if $D_j \searrow Z_j$, then it is possible to take $D_j = G_j$. The only difficulty with the construction above is that G_j may not retract into Z_j; however, by amalgamating the Z_j, this may be overcome since for each j there exists j' such that $G_{j'}$ retracts into Z_j. This concludes the proof of Theorem 4.1.

Corollary 4.3. *For $Z \in E^{k-1}$, there exists the exact sequence $0 \to S_1(Z \times R) \to S_0(Z) \to 0$, where $Z \times R$ is given the product k-fold end structure.*

Proof. Restrict attention to $Z \times R^+$, and define the Z_i of Theorem 4.1 as $Z_i = Z \times [i, \infty)$. It now follows from the translation properties of $Z \times R$ and the ability to collapse Z_i to Z that $\lim' S_1^b(Z_i) = 0$ and $\lim S_0^b(Z_i) = S_0^b(Z) = S_0(Z)$. Hence the exact sequence of Theorem 4.1 reduces to $0 \to S_1(Z \times R) \to S_0(Z) \to 0$.

5. Connectedness in E^k

An inclusion $(X, x) \to (Y, y)$ in the category E^k is defined to be *m-connected in* E^k if: (i) for $k = 0$, X and Y are connected, and $\pi_i(Y, X) = 0$ for $1 \le i \le m$; (ii) for $k \ge 1$, the pair (Y, X) satisfies (i) and additionally, for a given j, $0 \le j \le k$, $r_j, \dots, r_k > 0$, there exist $s_j, \dots s_k > 0$ such that if $y_i \circ f(D^m) \subset [0, r_i]$ for $f: (D^m, \partial D^m) \to (Y, X)$, $j+1 \le i \le k$, and m-disc D^m and $y_j \circ f(D^m) \subset [s_j, \infty)$, then there exists $f_t: (D^m, \partial D^m) \to (Y, X)$ such that $f_0 = f$, $f_1(D^m) \subset X$, $y_j \circ f_t(D^m) \subset [r_j, \infty)$ and $y_i \circ f_t(D^m) \subset [0, s_i]$ for $j+1 \le i \le k$ (note that the appropriate parts of the definition are interpreted vacuously for $j = 0$ or $j = k$). For example, if $k = 2$, then the definition breaks into the following cases:

(a) for $j = 0$, $r_1, r_2 > 0$ there exist $s_1, s_2 > 0$ such that if $y_i \circ f(D^m) \subset [0, r_i]$, $i = 1, 2$, then there exists $f_t: (D^m, \partial D^m) \to (Y, X)$ such that $f_0 = f$, $f_1(D^m) \subset X$, and $y_i \circ f_t(D^m) \subset [0, s_i]$, $i = 1, 2$;

(b) for $j = 1$, $r_1, r_2 > 0$ there exist $s_1, s_2 > 0$ such that if $y_2 \circ f(D^m) \subset [0, r_2]$ and $y_1 \circ f(D^m) \subset (s_1, \infty)$, then there exists f_t such that $f_0 = f$, $f_1(D^m) \subset X$, $y_1 \circ f_t(D^m) \subset [r_1, \infty)$ and $y_2 \circ f_t(D^m) \subset [0, s_2]$;

(c) for $j = 2$, $r_2 > 0$ there exists $s_2 > 0$ such that if $y_2 \circ f(D^m) \subset [s_2, \infty)$, then there exists f_t such that $f_0 = f$, $f_1(D^m) \subset X$, and $y_2 \circ f_t(D^m) \subset [r_2, \infty)$.

Standard propositions may also be generalized in the context of this definition.

Proposition 5.1. *Let $X \to Y$ be a homotopy equivalence in E^k. Then $X \to Y$ is m-connected in E^k for every m.*

Proposition 5.2. *If $(X, x) \to (Y, y)$ is m-connected in E^k for every m, then $X \to Y$ is a homotopy equivalence in E^k.*

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